for the I6 injector indicate a zone of high evaporation and combustion approximately 4 in. from the injector face. The injector with impingement distances less than 1 in. has a peak in the pressure gradient curve 6 in. from the injector face, whereas the pressure gradient for the last injector indicates a peak near the nozzle entrance. This implies a high degree of evaporation and gas generation in the downstream portions of the chamber.

Data have recently been obtained from three tangential sheet impinging injectors with design impingement points greater than 1 in. Although it cannot be readily determined whether or not impingement takes place at such large distances from the face, the data indicate that, as the intended impingement point is moved downstream, the rate of pressure drop decreases. At the maximum impingement distance tested, the effect of poorer mixing and low propellant utilization is evidenced by the decrease in over-all chamber pressure, despite a somewhat higher propellant flow rate.

The performance with this last injector (I9) is inferior to that obtained with the showerhead S2 injector discussed previously. Aside from the obvious differences between the two injectors, the showerhead contains four alternating rings of fuel and oxidizer, whereas the impinging injector contains a single ring of doublets. It appears then that distribution of the propellant over a greater portion of the chamber cross section enhances mixing and atomization and promotes the evaporation process. Tests conducted with a two-ring showerhead indicate that a chamber length of 21 in. is required for optimum performance, whereas the four-ring injector only required 14 in.

The three injectors just discussed are equipped with pressure taps in the injector face. The data show that the point of maximum chamber pressure is downstream from the injector face, and the pressure gradient near the face is positive. This is indicated by a negative value of $-(\Delta P/\Delta x)$. This suggests the existence of a recirculation zone or negative velocity gradient in the region near the injector face.

Conclusions

Variations in injector configuration produce variations in axial pressure history in a constant area liquid propellant rocket motor. These pressure variations are due to changes in the mixing and shattering characteristics of different injectors and are concurrent with unique evaporation and mass liberation profiles. For an injector that promotes rapid droplet breakup and intimate mixing of propellants, such as the I6 used in these experiments, the chamber pressure has a maximum gradient close to the injector face with a correspondingly high gas acceleration. This action exerts additional gasdynamic effects on the unconsumed droplets, enhancing propellant consumption.

Since the attenuation or amplification of a pressure wave is dependent on the gasdynamic field through which it propagates, the inherent stability or instability of an injector-chamber system is keyed to the pressure-velocity gradient produced.

In addition, the data can be used to establish design criteria for steady-state operation and to predict critical conditions that can exist because of large pressure gradients that result in high heat-transfer rates to the injector face or rocket chamber wall.

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Euler Angles for Libration Analysis

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INCREASINGLY stringent satellite attitude stability requirements have necessitated accurate dynamical descriptions of librational motion consisting of limited angular displacements about satellite centroid. Although Euler's angle convention has always been followed in treatises on dynamics of rotating bodies, it introduces a peculiar inconvenience in discussions of librations: small angular displacements of arbitrary direction cannot be represented by a set of small values of the Eulerian angles. Although quaternions or different angle conventions could be introduced which are not subject to this disadvantage, there is an understandable reluctance to adopt these, owing to the attendant loss of accessibility to the classic literature and methods which this would entail. It is shown below that sets of finite Euler angles can be found which possess all of the properties desired for treating general librational motions, thus obviating the necessity for introduction of unconventional analytical techniques.

The need for a convention for representing angular displacements is recalled to be a consequence of the fact that these are not vector quantities, since magnitude, direction, and also sequence determine the resultant of two or more rotations; i.e., angular displacements do not enjoy the commutative property of addition. Among the many ways of securing an arbitrary rotation by compounding rotations about a set of orthogonal directions, Euler's convention, being adequate, has long held universal acceptance. A rotation ψ about one axis X, followed by a rotation θ about a perpendicular axis Y, and a final rotation φ about the original axis bring the axes to positions x, y. The two right-handed triads X, Y, Z and x, y, z are then related by equations compactly expressed in matrix form[1]:

$$\begin{bmatrix} x \\ x \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\psi\sin\theta & -\cos\psi\sin\theta \\ \sin\theta\sin\varphi & \cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi & \sin\psi\cos\varphi + \cos\psi\cos\theta\sin\varphi \\ \sin\theta\cos\varphi & -\cos\psi\sin\varphi - \sin\psi\cos\theta\cos\varphi & -\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad (1)$$

Although zero displacements Ψ, θ, φ clearly do reduce the transformation matrix to unit diagonal form, it is also seen directly that first-order small values of these angles do not correspond to a rotation of arbitrary direction. For this purpose, it is sufficient to compare (1) with the transformation that corresponds to infinitesimal rotations α, β, γ about axes X, Y, Z, as done in librations analysis[2]:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad (2)$$

Note in particular that the matrix element $\sin\psi \sin\theta$ vanishes to the second order in the Eulerian transformation (1), whereas the corresponding Z rotation γ in (2) is only first-order small. Hence, small values of the Euler angles do not represent small rotations that have a Z component. The geometrical reason for this is to be found in the unsymmetrical character of Euler's convention: two rotations about X, one about Y, and none about Z. The more symmetrical rotation sequence that leads to the transformation (2), on the other hand, entails exactly one rotation about each of three axes and thus is of a different class from Euler's, the same being true of course when the angles α, β, γ are finite and arranged in a definite sequence.

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The complete generality of Euler's rotations nonetheless assures that there must be values of ψ, θ, φ which correspond to all rotations, including arbitrary infinitesimal rotations about three orthogonal directions. These are found by considering those sets of three Euler rotations which leave the triad invariant, the final configuration congruent to the initial. One of these is $\psi = \pi/2$, $\theta = \pi/2$, $\varphi = 0$, leading to the required infinitesimal rotation by means of the angles

$$\psi = (\pi/2) + \alpha \qquad \theta = (\pi/2) + \gamma \qquad \varphi = \beta \quad (3)$$

When the angles α, β, γ are small,

$$\alpha \ll 1 \qquad \beta \ll 1 \qquad \gamma \ll 1 \quad (4)$$

the substitution of (3) into (1) recovers (2). The suitability of angles (3) is also geometrically evident because the two rotations of approximate magnitude $\pi/2$ have the effect of separating each rotation axis from the others by an angular distance $\pi/2$. This is clearly necessary for the representation of a resultant small rotation of arbitrary direction, but it is precluded by the Euler sequence when ψ or θ is limited to small values, the remaining angle φ also being small.

The usefulness of a set of rotations such as (3) (other sets may also be found which serve equally well) is that accurate calculation of libration amplitudes, mode coupling, stability, and related characteristics requires consideration of both infinitesimal and finite values of α, β, γ in unrestricted combinations; angles (3) permit the partially stabilized libration to be joined to the completely stabilized libration in which $\alpha = \beta = \gamma = 0$. This is accomplished, moreover, within the orthodox framework of Euler rotations in a generality not possible with the small values of ψ, θ, φ, which correctly represent only the limiting state of zero libration.

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Stationary Earth Orbits

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As observed from the earth, the orbit of an earth satellite is a Keplerian ellipse on which is superposed a rotation equal in magnitude but opposite in sign to that of the earth. It is possible to remove this precession by a suitably programed continuous thrust, leading to an orbit that is "stationary" with respect to the earth.† The simplest example is a circular polar orbit constrained to remain in a plane of fixed longitude. Such orbits would have obvious advantages for a system of communication satellites.

Although the maintenance for long times of orbits that are stationary in the forementioned sense imposes requirements that are far beyond what is at present attainable, it seems worthwhile to sketch the very simple theory that establishes qualitatively what these requirements are.

Assuming that a reference frame in which the earth is at rest is an inertial frame, the equation of motion of a point satellite in a frame of reference rotating with the earth takes the well-known form

$$d^2\mathbf{r}/dt^2 = \mathbf{a}(\mathbf{r}) + \mathbf{S} - 2\boldsymbol{\omega} \times \mathbf{v} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

where **a** is the earth's gravitational acceleration at a point **r**, **S** is the acceleration produced by the programed thrust, **ω** the angular velocity of rotation of the earth, and **v** the velocity of the satellite relative to the earth. The last two terms are, respectively, the Coriolis and centrifugal accelerations introduced by the transformation to a coordinate system with rotation **ω**. A Keplerian ellipse stationary with reference to the earth is obtained by programing the thrust to give the acceleration‡:

$$\mathbf{S} = 2\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

It is readily shown that for low earth orbits the Coriolis term $2\boldsymbol{\omega} \times \mathbf{v}$ is predominant (more than 90% of the total acceleration out to five earth radii), and the order of magnitude of the average specific thrust $G = S/g$, where g is the acceleration of gravity at the surface of the earth, is given by $G = \omega v/g$. For $v = 26{,}000$ fps, that is, for orbital speed at the surface of the earth, $G = 0.06$. Although this is a relatively low specific thrust, the powerplant or propellant requirements for maintenance of a stationary orbit for long periods of time are prohibitively high. What these requirements are may be seen by considering the simple but typical case of a circular polar orbit at low altitude. The average over 1 rev of $|2\boldsymbol{\omega} \times \mathbf{v}|$ is then nearly equal to ωv, and the equation for propellant consumption may be written as $Ig(dm/dt) = -m\omega v$, where I is the specific impulse and v may be taken as constant. Integration of this equation leads immediately to $\ln(m_0/m_f) = (\omega v/Ig)\Delta t$, where Δt is the time in which the satellite mass is reduced from m_0 to m_f by propellant consumption. For an idealized situation in which the initial mass m_0 is composed of propellant, payload (mass m_L), and powerplant plus propulsion equipment (mass m_E), the equation foregoing for Δt can be put in the form

$$\Delta t = -(Ig/\omega v) \ln[l + (m_E/m_0)] \quad (1)$$

where $l = m_L/m_0$ is the payload ratio.

For the chemical propulsion, m_E/m_0 may be neglected to a first approximation if l is not too small. Equation (1) then becomes, for $I = 400$ sec and $G = \omega v/g = 0.06$ as for low earth orbit, $\Delta t = 6800 \ln l$ sec. For $l = 0.5$, Δt is of the order of the time of 1 rev of the satellite. If there should be a requirement for such a polar orbit that returns to the same point on the earth it could, however, be met more easily by a single impulsive change in velocity than by a continuous programed thrust.

Since Δt in Eq. (1) increases only logarithmically as l decreases but is proportional to I, longer times can best be attained by increasing the specific impulse. This suggests electrical propulsion, in which I can be made very large and programing of the thrust is relatively easy. However, the power required and accordingly the mass of the powerplant will also increase with I, and a maximum Δt for an optimum I is to be expected. That this is the case when the powerplant mass is proportional to the maximum jet power is easily seen analytically. Writing $m_E = bP$, where b is a constant and P the maximum power, leads at once to the relation $m_E/m_0 = \frac{1}{2}g^2bIG$. Inserting this in Eq. (1) and using the notation $x = l + \frac{1}{2}g^2bIG$ shows that $\Delta t \sim (x - l)\ln x$. The value x_0 of x which maximizes Δt for fixed l, b, and G can then be obtained by setting $(d/dx)(x - l)\ln x_0 = 0$. This results in the implicit equation $\ln x_0 = (l - x_0)/x_0$ for the determination of x_0. The maximized time Δt_m is then given by

$$\Delta t_m = \frac{2(x_0 - l)}{g^2G^2b} \ln\frac{1}{x_0} = \frac{5.8 \times 10^3(x_0 - l)}{b} \ln\frac{1}{x_0} \text{ sec} \quad (2)$$

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† It is the orbit that is stationary, not the position of the satellite as in the very special case of a 24-hr equatorial circular orbit.

‡ There is an amusing analogy to the Zeeman effect. The orbit of an atomic electron in a magnetic field **H** is stationary in a coordinate system having the Larmor precession $\boldsymbol{\omega}_L$ given by $(e/mc)\mathbf{v} \times \mathbf{H} = 2\boldsymbol{\omega}_L \times \mathbf{v}$, that is, one in which acceleration by the Lorentz force cancels the Coriolis acceleration.